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EXISTENCE OF THREE SOLUTIONS FOR EQUATIONS OF $p(x)$-LAPLACE TYPE

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ABSTRACT. We are concerned with the following elliptic equations with variable exponents

$$-\mathrm{div}(\varphi(x,\nabla u)) = \lambda f(x,u) + \lambda\theta g(x,u) \quad \text{in } \Omega,$$

which is subject to Dirichlet boundary condition. The purpose of this paper is to establish the existence of at least three solutions for the problem above by using as the main tool a variational principle due to Ricceri. In addition, we give information on size and location of an interval of λ's for which the given problem has either only the trivial solution or at least two nontrivial solutions.

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1. INTRODUCTION

In this paper, we consider the nonlinear elliptic equations of the $p(x)$-Laplace type

$$(\mathrm{B}_{\lambda,\theta}) \qquad \begin{cases} -\mathrm{div}(\varphi(x,\nabla u)) = \lambda f(x,u) + \lambda\theta g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function $\varphi(x,v)$ is of type $|v|^{p(x)-2}v$ with a continuous function $p:\overline{\Omega}\to(1,+\infty)$, $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions, and λ,θ are real parameters. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace type operator $\mathrm{div}(\varphi(x,\nabla u))$. We remember that the $p(x)$-Laplacian operator is defined by $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. The investigations for the $p(x)$-Laplace type problems have been widely studied by many researchers in various approaches; see [1, 12, 16, 17, 18, 19] and references therein.

Recently, Ricceri's critical point theorem [22] has been applied with success in several problems involving differential equations of variational type; see [1, 5, 17] and references therein. Liu and Shi [17] studied the existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x),q(x))$-Laplacian with Dirichlet boundary condition. The author in [1] obtained some existence and multiplicity results for nonlinear elliptic equations of the $p(x)$-Laplace operator in the whole space $\mathbb{R}^N$. The first aim of

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this paper is to establish the existence of at least three solutions for problem $(\mathrm{B}_{\lambda,\theta})$ by using as the main tool a variational principle due to Ricceri [22].

We point out that three-critical-points theorems introduced by Ricceri [21, 22] gave no further information on the size and location of a three critical points interval. However, the authors in [5] localized the interval for the existence of three solutions for homogeneous Dirichlet problem and inhomogeneous nonlinear Robin problem associated to the p-Laplace type operators which was motivated by the study of Arcoya and Carmona [2]. In particular, under suitable assumptions, to obtain the three critical points interval for the given problem in [5], they consider the first eigenvalue λ_1 of the p-Laplacian eigenvalue problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)\,|u|^{p-2}\,u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, λ_1 is defined by the Rayleigh quotient

$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega),u\neq 0} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega a(x)\,|u|^p\,dx}.$$

The positivity of λ_1 plays a decisive role in determining the three critical points interval in [5]. In contrast with the p-Laplacian eigenvalue problem, the infimum of all eigenvalues for the $p(x)$-Laplacian might be zero; see [11]. To overcome this difficulty, under appropriate condition for the variable exponent $p(\cdot)$, we give that the infimum of all eigenvalues for the $p(x)$-Laplacian is positive. Using this fact, the second goal of this paper is to determine precisely the intervals of λ's for which problem $(\mathrm{B}_{\lambda,\theta})$ has only the trivial solution and for which problem $(\mathrm{B}_{\lambda,\theta})$ admits at least two nontrivial solutions. To the best of our knowledge, the results on the localization of the interval for the existence of three solutions to equations of the $p(x)$-Laplacian are rare. This is novelty of the present paper.

To make a self-contained paper, we recall some definitions of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the variable exponent Lebesgue-Sobolev space $W^{1,p(\cdot)}(\Omega)$ which will be treated in the next section.

Set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1 \right\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) := \left\{ u : u \text{ is a measurable function, } \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

The dual space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

where the norm is

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}. \tag{1.1}$$

To illustrate the density of smooth functions in $W^{1,p(\cdot)}(\Omega)$, we need a definition of the log-Hölder continuity condition for the variable exponent p, namely, a function $p : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω if there is a constant C_0 such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x-y|} \tag{1.2}$$

for every $x, y \in \Omega$ with $|x-y| \leq 1/2$. As established in [6, 7], if Ω is a bounded domain with Lipschitz boundary and p satisfies the log-Hölder continuity condition, then smooth functions are dense in variable exponent Sobolev spaces.

This paper is organized as follows. We first state some preliminary lemmas and present some properties of the integral operators corresponding to problem $(\mathrm{B}_{\lambda,\theta})$. And then we will prove the existence of at least three solutions for problem $(\mathrm{B}_{\lambda,\theta})$ and localization of the intervals of λ's for which problem $(\mathrm{B}_{\lambda,\theta})$ has only the trivial solution and for which problem $(\mathrm{B}_{\lambda,\theta})$ admits at least two nontrivial solutions.

2. Preliminaries and Main Results

In this section, we briefly introduce some definitions and basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the variable exponent Lebesgue-Sobolev space $W^{1,p(\cdot)}(\Omega)$. For a deeper treatment on these spaces, we refer to [4, 6, 7, 8, 9, 10, 14].

Lemma 2.1. ([10]) *The space $L^{p(\cdot)}(\Omega)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $1/p(x)+1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Lemma 2.2. ([10]) *Denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx, \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

Then

(1) $\rho(u) > 1$ $(= 1;\ < 1)$ *if and only if* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$ $(= 1;\ < 1)$, *respectively;*
(2) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, *then* $\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}$;
(3) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, *then* $\|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}$.

Lemma 2.3. ([7]) *Let $q \in L^\infty(\Omega)$ be such that $1 \leq p(x)q(x) \leq \infty$ for almost all $x \in \Omega$. If $u \in L^{q(\cdot)}(\Omega)$ with $u \neq 0$, then*

(1) *if* $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} > 1$, *then*
$\|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \|\,|u|^{q(x)}\,\|_{L^{p(\cdot)}(\Omega)} \le \|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\Omega)}$;
(2) *if* $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} < 1$, *then*
$\|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \|\,|u|^{q(x)}\,\|_{L^{p(\cdot)}(\Omega)} \le \|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\Omega)}$.

Lemma 2.4. ([6]) *Let* $p \in C_+(\overline{\Omega})$ *satisfy the log-Hölder continuity condition. For* $u \in W_0^{1,p(\cdot)}(\Omega)$, *the* $p(\cdot)$*-Poincaré inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds, where the positive constant C *depends on* p *and* Ω.

Lemma 2.5. ([10]) *Let* $\Omega \subset \mathbb{R}^N$ *be an open, bounded set with Lipschitz boundary and let* $p \in C_+(\overline{\Omega})$ *satisfy the log-Hölder continuity condition with* $1 < p_- \le p_+ < \infty$. *If* $q \in L^\infty(\Omega)$ *with* $q_- > 1$ *satisfies*

$$q(x) \le p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x), \end{cases}$$

then we have a continuous imbedding

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

and the imbedding is compact if $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0$.

In what follows, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition and let us define our basic space $X := W_0^{1,p(\cdot)}(\Omega)$ with the norm

$$\|u\|_X = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

which is equivalent to the norm (1.1) due to Lemma 2.4. Furthermore, $\langle\cdot,\cdot\rangle$ denotes the pairing of X and its dual X^*.

Definition 2.6. *We say that* $u \in X$ *is a weak solution of problem* $(\mathrm{B}_{\lambda,\theta})$ *if*

$$\int_\Omega \varphi(x,\nabla u)\cdot\nabla v\,dx = \lambda\int_\Omega f(x,u)v\,dx + \lambda\theta\int_\Omega g(x,u)v\,dx$$

for all $v \in X$.

We assume that $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with the continuous derivative with respect to v of the mapping $\Phi_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x,v)$, that is, $\varphi(x,v) = \frac{d}{dv}\Phi_0(x,v)$. Suppose that φ and Φ_0 satisfy the following assumptions: For $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$,

(J1) the following equality

$$\Phi_0(x,\mathbf{0}) = 0$$

holds for almost all $x \in \Omega$.

(J2) there is a nonnegative constant d such that

$$|\varphi(x,v)| \le d|v|^{p(x)-1}$$

holds for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$.

(J3) $\Phi_0(x,\cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \Omega$.

(J4) the following relation

$$c_*|v|^{p(x)} \le \varphi(x,v)\cdot v \le p_+\Phi_0(x,v)$$

holds for all $x \in \Omega$ and $v \in \mathbb{R}^N$, where c_* is a positive constant.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \Phi_0(x, \nabla u)\, dx.$$

Under assumptions (J1), (J2) and (J4), it follows from [16, Lemma 3.2] that the functional Φ is well-defined on X, $\Phi \in C^1(X,\mathbb{R})$ and its Gâteaux derivative is given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \varphi(x,\nabla u)\cdot \nabla v\, dx.$$

Next adopting an argument given in the proof of Theorem 4.1 of [15], we give that the operator Φ' is a mapping of type (S_+) which plays an important role in obtaining main results.

Lemma 2.7. *Assume that* (J1)–(J4) *hold. Then the functional* $\Phi : X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X*. Moreover, the operator* Φ' *is a mapping of type* (S_+)*, i.e., if* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$ *and* $\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \le 0$*, then* $u_n \to u$ *in* X *as* $n \to \infty$*.*

Proof. Assumption (J3) implies that the functional Φ is convex and weakly lower semicontinuous on X. Moreover, proceeding the analogous argument as in the proof of Theorem 4.1 in [15], it is immediate that the operator Φ' is a mapping of type (S_+). □

Corollary 2.8. *Assume that* (J1)–(J4) *hold. Then the operator* $\Phi' : X \to X^*$ *is homeomorphism onto* X^**.*

Proof. It is obvious that the operator Φ' is strictly monotone, coercive and hemicontinuous on X. By the Browder-Minty theorem, the inverse operator $(\Phi')^{-1}$ exists; see Theorem 26.A in [24]. The proof of continuity of the inverse operator $(\Phi')^{-1}$ is analogous to that in the case of a constant exponent and hence omit it here. □

Before dealing with our main results, we need the following assumptions on f and g. Let us put $F(x,t) = \int_0^t f(x,s)\,ds$ and $G(x,t) = \int_0^t g(x,s)\,ds$. Then we assume that

(H1) $p \in C_+(\overline{\Omega})$ and $1 < p_- \le p_+ < p^*(x)$ for all $x \in \Omega$.

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two nonnegative functions $a_1, b_1 \in L^\infty(\Omega)$ such that

$$|f(x,s)| \le a_1(x) + b_1(x)\,|s|^{\gamma_1(x)-1}$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where $\gamma_1 \in C_+(\overline{\Omega})$ and $(\gamma_1)_+ < p_-$.

(F2) There exist an element x_1 in Ω, a real number s_1 and a positive constant r_1 so small that

$$\int_{B_N(x_1,r_1)} F(x,|s_1|)\,dx > 0 \qquad \text{and} \qquad F(x,t) \geq 0$$

for almost all $x \in B_N(x_1,r_1) \setminus B_N(x_1,\sigma r_1)$ with $\sigma \in (0,1)$ and for all $0 \leq t \leq |s_1|$, where $B_N(x_1,r_1) = \{x \in \Omega : |x - x_1| \leq r_1\} \subseteq \Omega$.

(G1) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two nonnegative functions $a_2, b_2 \in L^\infty(\Omega)$ such that

$$|g(x,s)| \leq a_2(x) + b_2(x)\,|s|^{\gamma_2(x)-1}$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where $\gamma_2 \in C_+(\overline{\Omega})$ and $(\gamma_2)_+ < p_-$.

Then we define the functionals $\Psi, H : X \to \mathbb{R}$ by

$$\Psi(u) = -\int_\Omega F(x,u)\,dx \quad \text{and} \quad H(u) = -\int_\Omega G(x,u)\,dx.$$

It is easy to check that $\Psi, H \in C^1(X,\mathbb{R})$ and these Gâteaux derivatives are

$$\langle \Psi'(u), v\rangle = -\int_\Omega f(x,u)v\,dx \quad \text{and} \quad \langle H'(u), v\rangle = -\int_\Omega g(x,u)v\,dx.$$

for any $u, v \in X$; see [8].

Lemma 2.9. ([22]) *Let X be a reflexive real Banach space; $I \subset \mathbb{R}$ an interval; $\Phi : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1-functional whose derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a C^1-functional with compact derivative. In addition, the functional Φ is bounded on each bounded subset of X. Assume that*

$$\lim_{\|u\|_X \to \infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in I$ and there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{u \in X} (\Phi(u) + \lambda(\Psi(u) + \rho)) < \inf_{u \in X} \sup_{\lambda \in I} (\Phi(u) + \lambda(\Psi(u) + \rho)). \tag{2.1}$$

Then there exist a nonempty open set $\Lambda \subset I$ and a positive real number $R > 0$ with the following property: for every $\lambda \in \Lambda$ and every C^1-functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\theta \in [0,\delta]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \theta J'(u) = 0$$

has at least three solutions in X whose norms are less than R.

Lemma 2.10. *Assume that* (J1), (J2), (J4), (H1), (F1), *and* (G1) *hold. Then*

$$\lim_{\|u\|_X \to \infty} \{\Phi(u) + \lambda\,(\Psi(u) + \theta H(u))\} = +\infty$$

for all $\lambda, \theta \in \mathbb{R}$.

Proof. For $\|u\|_X$ large enough and for all $\lambda, \theta \in \mathbb{R}$, it follows from Lemmas 2.1, 2.2, 2.3, and 2.5 that

$$\begin{aligned}
&\Phi(u) + \lambda\,(\Psi(u) + \theta H(u)) \\
&\quad = \int_\Omega \Phi_0(x,\nabla u)\,dx - \lambda \int_\Omega F(x,u)\,dx - \lambda\theta \int_\Omega G(x,u)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{c_*}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx - |\lambda|\int_\Omega |a_1(x)|\,|u|\,dx - |\lambda|\int_\Omega \frac{1}{\gamma_1(x)}\,|b_1(x)|\,|u|^{\gamma_1(x)}\,dx\\
&\qquad - |\lambda|\,|\theta|\int_\Omega |a_2(x)|\,|u|\,dx - |\lambda|\,|\theta|\int_\Omega \frac{1}{\gamma_2(x)}\,|b_2(x)|\,|u|^{\gamma_2(x)}\,dx\\
&\geq \frac{c_*}{p_+}\|\nabla u\|^{p_-}_{L^{p(\cdot)}(\Omega)} - |\lambda|\,\|a_1\|_{L^\infty(\Omega)}\|u\|_{L^1(\Omega)} - \frac{|\lambda|}{(\gamma_1)_-}\|b_1\|_{L^\infty(\Omega)}\|\,|u|^{\gamma_1(x)}\,\|_{L^1(\Omega)}\\
&\qquad - |\lambda|\,|\theta|\,\|a_2\|_{L^\infty(\Omega)}\|u\|_{L^1(\Omega)} - \frac{|\lambda|\,|\theta|}{(\gamma_2)_-}\|b_2\|_{L^\infty(\Omega)}\|\,|u|^{\gamma_2(x)}\,\|_{L^1(\Omega)}\\
&\geq \frac{c_*}{p_+}\|u\|^{p_-}_X - |\lambda|\,C_1\|a_1\|_{L^\infty(\Omega)}\|u\|_X - \frac{|\lambda|}{(\gamma_1)_-}\|b_1\|_{L^\infty(\Omega)}\|u\|^{(\gamma_1)_+}_{L^{\gamma_1(\cdot)}(\Omega)}\\
&\qquad - |\lambda|\,|\theta|\,C_2\|a_2\|_{L^\infty(\Omega)}\|u\|_X - \frac{|\lambda|\,|\theta|}{(\gamma_2)_-}\|b_2\|_{L^\infty(\Omega)}\|u\|^{(\gamma_2)_+}_{L^{\gamma_2(\cdot)}(\Omega)}\\
&\geq \frac{c_*}{p_+}\|u\|^{p_-}_X - |\lambda|\,C_1\|a_1\|_{L^\infty(\Omega)}\|u\|_X - \frac{|\lambda|\,C_3}{(\gamma_1)_-}\|b_1\|_{L^\infty(\Omega)}\|u\|^{(\gamma_1)_+}_X\\
&\qquad - |\lambda|\,|\theta|\,C_2\|a_2\|_{L^\infty(\Omega)}\|u\|_X - \frac{|\lambda|\,|\theta|\,C_4}{(\gamma_2)_-}\|b_2\|_{L^\infty(\Omega)}\|u\|^{(\gamma_2)_+}_X
\end{aligned}
$$

for some positive constants C_1, C_2, C_3 and C_4. Since $p_- > (\gamma_1)_+$ and $p_- > (\gamma_2)_+$, we deduce that

$$\lim_{\|u\|_X\to\infty}\{\Phi(u) + \lambda\,(\Psi(u) + \theta H(u))\} = +\infty$$

for all $\lambda, \theta \in \mathbb{R}$. □

The following lemma plays a key role in obtaining the remaining assumption (2.1) of Lemma 2.9.

Lemma 2.11. ([20]) Let X be a nonempty set and Φ, Ψ two real functionals on X. Assume that there are $\mu > 0$ and $u_0, u_1 \in X$ such that

$$\Phi(u_0) = -\Psi(u_0) = 0, \quad \Phi(u_1) > \mu,$$

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])} -\Psi(u) < \mu\left(-\frac{\Psi(u_1)}{\Phi(u_1)}\right). \tag{2.2}$$

Then, for each ρ satisfying

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])} -\Psi(u) < \rho < \mu\left(-\frac{\Psi(u_1)}{\Phi(u_1)}\right),$$

one has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho + \Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho + \Psi(u)))\,.$$

Employing Lemma 2.9 with Lemma 2.11, we establish the existence at least three solutions for problem $(\mathrm{B}_{\lambda,\theta})$.

Theorem 2.12. *Assume that* (J1)–(J4), (H1), (F1)–(F2), *and* (G1) *hold. Moreover, assume that*

(F3) $\limsup_{s\to 0}\left(\operatorname{ess\,sup}_{x\in\Omega}\frac{|F(x,s)|}{|s|^{\kappa(x)}}\right) < +\infty$, *where* $\kappa \in C_+(\overline{\Omega})$ *satisfies* $p_+ < \kappa_- < \kappa(x) < p^*(x)$ *for all* $x \in \Omega$.

Then there exist a nonempty open set $\Lambda \subset [0, +\infty)$ and a positive real number $R > 0$ with the following property: for every $\lambda \in \Lambda$, there exists $\delta > 0$ such that for each $\theta \in [0, \delta]$, problem $(\mathrm{B}_{\lambda,\theta})$ has at least three solutions in X whose norms are less then R.

Proof. It follows from Lemma 2.7 that the functional $\Phi : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous C^1-functional. Moreover, it is bounded on each bounded subset of X. By Corollary 2.8, there exists a continuous inverse operator $(\Phi')^{-1} : X^* \to X$. From Lemma 2.5, the modification of the proof of Lemma 3.5 in [12] shows that the operator $\Psi' : X \to X^*$ is compact. From Lemma 2.10 with $\theta = 0$, we know that

$$\lim_{\|u\|_X \to \infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $u \in X$ and for all $\lambda \in \mathbb{R}$.

To show all assumptions in Lemma 2.9, we verify the assumption (2.1). Let $s_1 \neq 0$ be from (F2). For $\sigma \in (0, 1)$, define

$$u_\sigma(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B_N(x_1, r_1) \\ |s_1| & \text{if } x \in B_N(x_1, \sigma r_1) \\ \frac{|s_1|}{r_1(1-\sigma)}(r_1 - |x - x_1|) & \text{if } x \in B_N(x_1, r_1) \setminus B_N(x_1, \sigma r_1). \end{cases} \tag{2.3}$$

It is obvious that $0 \le u_\sigma(x) \le |s_1|$ for all $x \in \Omega$ and $u_\sigma \in X$. Moreover, the fact that $L^{p(\cdot)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ implies

$$\begin{aligned} \|u_\sigma\|_X^\alpha = \|\nabla u_\sigma\|_{L^{p(\cdot)}(\Omega)}^\alpha &\ge C_5 \int_\Omega |\nabla u_\sigma|^{p_-}\, dx \\ &= \frac{C_5\, |s_1|^{p_-} (1 - \sigma^N)}{(1-\sigma)^{p_-}} r_1^{N-p_-} \omega_N > 0 \end{aligned}$$

for a positive constant C_5, where α is either p_+ or p_- and ω_N is the volume of $B_N(0, 1)$. Hence we have

$$\begin{aligned} -\Psi(u_\sigma) = &\int_{B_N(x_1,\sigma r_1)} F(x, |s_1|)\, dx \\ &+ \int_{B_N(x_1,r_1)\setminus B_N(x_1,\sigma r_1)} F\left(x, \frac{|s_1|}{r_1(1-\sigma)}(r_1 - |x - x_1|)\right) dx \\ > 0. \end{aligned}$$

From condition (F3), there exist $\eta \in (0, 1]$ and a positive constant C_6 such that

$$F(x, s) < C_6\, |s|^{\kappa(x)} < C_6\, |s|^{\kappa_-} \tag{2.4}$$

for almost all $x \in \Omega$ and for all $s \in [-\eta, \eta]$. Consider two positive constants M_1 and M_2 given by

$$M_1 = \sup_{|s|>1} \frac{C(|s| + |s|^{(\gamma_1)_+})}{|s|^{\kappa_-}} \quad \text{and} \quad M_2 = \sup_{\eta<|s|<1} \frac{C(|s| + |s|^{(\gamma_1)_-})}{|s|^{\kappa_-}}$$

for a positive constant C. Then it follows from (2.4) and (F1) that

$$F(x, s) < M\, |s|^{\kappa_-}$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$, where $M = \max\{C_6, M_1, M_2\}$. Fix a real number μ such that $0 < \mu < 1$. If μ satisfies $(c_*/p_+)\,\|u\|_X^{p_+} \le \mu < 1$, where c_* is the positive constant from (J4), then we assert

$$-\Psi(u) = \int_\Omega F(x,u)\,dx < M\int_\Omega |u|^{\kappa_-}\,dx \le C_7\|u\|_X^{\kappa_-} \le C_8\mu^{\frac{\kappa_-}{p_+}} \tag{2.5}$$

for some positive constants C_7 and C_8. Since $\kappa_- > p_+$, the relation (2.5) implies that

$$\lim_{\mu\to 0+} \frac{\sup_{\frac{c_*}{p_+}\|u\|_X^{p_+}\le\mu} -\Psi(u)}{\mu} = 0. \tag{2.6}$$

Now we check all assumptions in Lemma 2.11. Let us fix a real number μ_0 such that

$$0 < \mu < \mu_0 \le \frac{c_*}{p_+}\min\left\{\|u_\sigma\|_X^{p_+}, \|u_\sigma\|_X^{p_-}, 1\right\} \le \frac{c_*}{p_+},$$

where u_σ was defined in (2.3). By Lemmas 2.2, 2.5, and assumption (J4), we have

$$\Phi(u_\sigma) = \int_\Omega \Phi_0(x,\nabla u_\sigma)\,dx \ge \int_\Omega \frac{c_*}{p_+}|\nabla u_\sigma|^{p(x)}\,dx \ge \frac{c_*}{p_+}\|u_\sigma\|_X^{p_+} \ge \mu_0 > \mu$$

for $\|u_\sigma\|_X < 1$ and

$$\Phi(u_\sigma) = \int_\Omega \Phi_0(x,\nabla u_\sigma)\,dx \ge \int_\Omega \frac{c_*}{p_+}|\nabla u_\sigma|^{p(x)}\,dx \ge \frac{c_*}{p_+}\|u_\sigma\|_X^{p_-} \ge \mu_0 > \mu$$

for $\|u_\sigma\|_X > 1$. Relation (2.6) implies that

$$\sup_{\frac{c_*}{p_+}\|u\|_X^{p_+}\le\mu} -\Psi(u) \le \frac{\mu}{2}\left(-\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)}\right) < \mu\left(-\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)}\right).$$

For any $u \in \Phi^{-1}((-\infty,\mu])$, it is immediate that $\Phi(u) \le \mu$ and then

$$\frac{c_*}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx \le \int_\Omega \Phi_0(x,\nabla u)\,dx \le \mu.$$

Hence we deduce that

$$\int_\Omega |\nabla u|^{p(x)}\,dx \le \frac{p_+}{c_*}\mu < \frac{p_+}{c_*}\mu_0 < 1.$$

By the inequality above and Lemma 2.2, we assert $\|u\|_X < 1$. It follows that

$$\frac{c_*}{p_+}\|u\|_X^{p_+} \le \int_\Omega \Phi_0(x,\nabla u)\,dx \le \mu.$$

Then we have that

$$\Phi^{-1}((-\infty,\mu]) \subset \left\{u \in X : \frac{c_*}{p_+}\|u\|_X^{p_+} \le \mu\right\}.$$

This implies that

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])} -\Psi(u) \le \sup_{\frac{c_*}{p_+}\|u\|_X^{p_+}\le\mu} -\Psi(u) < \mu\left(-\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)}\right),$$

that is,

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])} -\Psi(u) < \mu\left(-\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)}\right).$$

Thus we can choose $\mu > 0, u_0 = 0$, and $u_1 = u_\sigma$ such that relations $\Phi(u_\sigma) > \mu$ and (2.2) are satisfied. Also there exists a real number ρ such that

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])} -\Psi(u) < \rho < \mu\left(-\frac{\Psi(u_\sigma)}{\Phi(u_\sigma)}\right).$$

Set $I = [0, +\infty)$. Due to Lemma 2.11, we obtain that

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\Psi(u) + \rho)) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\Psi(u) + \rho)).$$

Define the functional $J : X \to \mathbb{R}$ by $J = \lambda H$. Since it is clear that the functional J is C^1-functional with compact derivative, the functionals Φ, Ψ, and J satisfy all assumptions of Lemma 2.9. This completes the proof. □

It is well known that Theorem 2.12 gives no further information on the size and location of the open set Λ. Hence we localize the interval for the existence of at least three solutions for problem $(\mathrm{B}_{\lambda,\theta})$ by applying the three-critical-points theorem in [2]. To do this, we consider the following eigenvalue problem

$$\text{(E)} \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda m(x)\,|u|^{p(x)-2}\,u & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

The positivity of the infimum of all eigenvalues for problem (E) is important to assert our main result. The proof of the following lemma is analogous to that of Proposition 3.7 in [13].

Lemma 2.13. *Assume that* (H1) *holds. Moreover, suppose that*

(H2) $m \in L^\infty(\Omega)$ *and* $m(x) > 0$ *for almost all* $x \in \Omega$.

Denote the quantity

$$\lambda_* = \inf_{u\in X\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega m(x)\,|u|^{p(x)}\,dx}. \tag{2.7}$$

Then there is $u_1 \in X$ *with* $\int_\Omega m(x)\,|u_1|^{p(x)}\,dx = 1$ *such that the infimum* λ_* *in* (2.7) *will be attained and* u_1 *represents an eigenfunction for problem* (E) *corresponding to* λ_*, *that is,* λ_* *is a positive eigenvalue of problem* (E). *In particular,*

$$\lambda_*\int_\Omega m(x)\,|u_1|^{p(x)}\,dx \leq \int_\Omega |\nabla u|^{p(x)}\,dx$$

for every $u \in X$.

Proof. It follows from Lemmas 2.4, 2.5, and assumption (H2) that λ_* is a positive number. Let us denote the functionals $\tilde\Phi, \tilde\Psi : X \to \mathbb{R}$ by

$$\tilde\Phi(u) = \int_\Omega |\nabla u|^{p(x)}\,dx \qquad \text{and} \qquad \tilde\Psi(u) = \int_\Omega m(x)\,|u|^{p(x)}\,dx$$

for any $u \in X$. Then it is easy to show that $\tilde\Phi$ and $\tilde\Psi$ are continuously Gâteaux differentiable, convex in X, and $\tilde\Phi'(0) = \tilde\Psi'(0) = 0$. From Lemma 2.7, $\tilde\Phi$ is weakly lower semicontinuous on X. The convexity of $\tilde\Psi$ implies that $\tilde\Psi$ is also weakly lower semicontinuous on X. Since any C^1-functional on X with compact derivative is sequentially weakly continuous on X, $\tilde\Psi$ is sequentially weakly continuous on X; see Corollary 41.9 in [23]. It is clear that $\tilde{\Phi}$ is coercive in X. By using contradiction argument, we assert $\tilde{\Phi}$ is coercive in $\{u \in X : \tilde{\Psi}(u) \le 1\}$. In conclusion, all the assumption of Theorem 6.3.2 in [3] are fulfilled and so λ_* is achieved in $\{u \in X : \tilde{\Psi}(u) = 1\}$. Namely, there exists an element $u_1 \in X$ with $\int_\Omega m(x)\,|u_1|^{p(x)}\,dx = 1$, which realizes the infimum in (2.7) and represents an eigenfunction for problem (E) corresponding to λ_*. This completes the proof. □

In the rest of this paper, we localize precisely the intervals of λ's for which problem $(\mathrm{B}_{\lambda,\theta})$ has either only the trivial solution or at least two nontrivial solutions. To do this, we assume that

(F4) $\limsup_{s\to 0} \frac{|f(x,s)|}{m(x)|s|^{\xi_1(x)-1}} < +\infty$ uniformly for almost all $x \in \Omega$, where $\xi_1 \in C_+(\overline{\Omega})$ with $p(x) < \xi_1(x) < p^*(x)$ for all $x \in \Omega$.

(G2) $\limsup_{s\to 0} \frac{|g(x,s)|}{m(x)|s|^{\xi_2(x)-1}} < +\infty$ uniformly for almost all $x \in \Omega$, where $\xi_2 \in C_+(\overline{\Omega})$ with $p(x) < \xi_2(x) < p^*(x)$ for all $x \in \Omega$.

Let us introduce two functions

$$\chi_1(r) = \inf_{u\in\Psi^{-1}((-\infty,r))} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r}, \tag{2.8}$$

$$\chi_2(r) = \sup_{u\in\Psi^{-1}((r,+\infty))} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \tag{2.9}$$

for every $r \in (\inf_{u\in X}\Psi(u), \sup_{u\in X}\Psi(u))$. Denote the crucial values

$$\mathcal{C}_f = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{m(x)\,|s|^{p(x)-1}} \quad \text{and} \quad \mathcal{C}_g = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|g(x,s)|}{m(x)\,|s|^{p(x)-1}}.$$

Then the same arguments in [5] imply that $\mathcal{C}_f$ and $\mathcal{C}_g$ are well defined, positive constants, and furthermore the following relations hold;

$$\operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|F(x,s)|}{m(x)\,|s|^{p(x)}} = \frac{\mathcal{C}_f}{p_-} \quad \text{and} \quad \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|G(x,s)|}{m(x)\,|s|^{p(x)}} = \frac{\mathcal{C}_g}{p_-}. \tag{2.10}$$

The next result represents the differentiable version of the Arcoya and Carmona Theorem 3.10 in [2].

Lemma 2.14. *Let Φ, Ψ be two functionals on X such that weakly lower semicontinuous and continuously Gâteaux differentiable in X. Let Ψ be nonconstant and H be continuously Gâteaux differentiable with compact derivative H'. Let also $\Phi' : X \to X^*$ be a mapping of type (S_+) and Ψ' be a compact operator. Assume that there exist an interval $I \subset \mathbb{R}$ and a number $\tau > 0$ such that for every $\lambda \in I$ and every $\theta \in [-\tau,\tau]$ the functional $I_{\lambda,\theta} = \Phi + \lambda(\Psi + \theta H)$ is coercive in X. If there exists*

$$r \in \left(\inf_{u\in X}\Psi(u), \sup_{u\in X}\Psi(u)\right) \quad \textit{such that} \quad \chi_1(r) < \chi_2(r) \tag{2.11}$$

and $(\chi_1(r),\chi_2(r)) \cap I \neq \emptyset$, then for every compact interval $[a,b]$ with $[a,b] \subset (\chi_1(r),\chi_2(r)) \cap I$, there exists $\gamma \in (0,\tau)$ with $|\theta| < \gamma$ such that the functional $I_{\lambda,\theta}$ admits at least three critical points for every $\lambda \in [a,b]$.

By applying Lemma 2.14, we can obtain the following assertion.

Theorem 2.15. *Assume* (J1)–(J4), (H1)–(H2), (F1)–(F2), (F4) *and* (G1)–(G2) *hold. Then we have*

(i) *for every* $\theta \in \mathbb{R}$ *there exists* $\ell_* = c_*\lambda_* p_-/p_+(\mathcal{C}_f + |\theta|\,\mathcal{C}_g)$ *such that problem* $(\mathrm{B}_{\lambda,\theta})$ *has only the trivial solution for all* $\lambda \in [0, \ell_*)$, *where* c_* *is a positive constant from* (J4) *and* λ_* *is a positive real number in* (2.7).

(ii) *if furthermore* f *satisfies the following assumption*

(F5) $\int_\Omega F(x, u_1(x))dx > d/c_* p_-$ *holds , where* u_1 *is the eigenfunction corresponding to the principle eigenvalue of problem* (E) *satisfying* $\int_\Omega m(x)\,|u_1|^{p(x)}\,dx = 1$ *and* $d, c_* > 0$ *are constants given in* (J2) *and* (J4), *respectively,*

then there exists $\tau > 0$ *such that problem* $(\mathrm{B}_{\lambda,\theta})$ *has at least two distinct nontrivial solutions for each compact interval* $[a, b] \subset (\ell^*,\ c_*\lambda_*)$, *where* $\ell^* = \chi_1(0) < c_*\lambda_*$ *with* $\ell^* \geq \ell_*$ *and for every* $\lambda \in [a, b]$ *and* $\theta \in (-\tau, \tau)$.

Proof. Under assumptions (J1)–(J4), (H1), (F1)–(F2), and (G1), all of the assumptions in Lemma 2.14 except the condition (2.11) are satisfied.

Now we prove the assertion (i). Let $u \in X$ be a nontrivial weak solution of problem $(\mathrm{B}_{\lambda,\theta})$. Then it is clear that

$$\int_\Omega \varphi(x, \nabla u) \cdot \nabla v\, dx = \lambda \int_\Omega f(x, u)v\, dx + \lambda\theta \int_\Omega g(x, u)v\, dx$$

for all $v \in X$. If we put $v = u$, then it follows from assumption (J4) and the definitions of $\mathcal{C}_f$ and $\mathcal{C}_g$ that

$$\begin{aligned}
c_*\lambda_* \int_\Omega |\nabla u|^{p(x)}\, dx \leq& \lambda_* \int_\Omega \varphi(x, \nabla u) \cdot \nabla u\, dx \\
=& \lambda_*\lambda \left(\int_\Omega f(x, u)u\, dx + \theta \int_\Omega g(x, u)u\, dx \right) \\
\leq& \lambda_*\lambda \Big(\int_\Omega \frac{f(x, u)}{m(x)\,|u|^{p(x)-1}} m(x)\,|u|^{p(x)}\, dx \\
&\qquad + \theta \int_\Omega \frac{g(x, u)}{m(x)\,|u|^{p(x)-1}} m(x)\,|u|^{p(x)}\, dx \Big) \\
\leq& \lambda_*\lambda(\mathcal{C}_f + |\theta|\,\mathcal{C}_g) \int_\Omega m(x)\,|u|^{p(x)}\, dx \\
\leq& \lambda(\mathcal{C}_f + |\theta|\,\mathcal{C}_g) \int_\Omega |\nabla u|^{p(x)}\, dx \\
\leq& \frac{\lambda(\mathcal{C}_f + |\theta|\,\mathcal{C}_g)p_+}{p_-} \int_\Omega |\nabla u|^{p(x)}\, dx.
\end{aligned}$$

Thus if u is a nontrivial weak solution of problem $(\mathrm{B}_{\lambda,\theta})$, then necessarily $\lambda \geq \ell_* = c_*\lambda_* p_-/p_+(\mathcal{C}_f + |\theta|\,\mathcal{C}_g)$, as claimed.

Next we show the assertion (ii). It is obvious that the crucial positive number

$$\ell^* = \chi_1(0) = \inf_{u \in \Psi^{-1}((-\infty,0))} \left(-\frac{\Phi(u)}{\Psi(u)} \right)$$

is well defined by assumption (F5). It follows from the definition of u_1 and (F5) that

$$\begin{aligned}\ell^* = \chi_1(0) &= \inf_{u\in\Psi^{-1}((-\infty,0))}\left(-\frac{\Phi(u)}{\Psi(u)}\right) \le -\frac{\Phi(u_1)}{\Psi(u_1)}\\ &= \frac{\int_\Omega \Phi_0(x,\nabla u_1)\,dx}{\int_\Omega F(x,u_1)\,dx} \le \frac{c_* p_-}{d}\int_\Omega \frac{d}{p(x)}\,|\nabla u_1|^{p(x)}\,dx \le c_*\lambda_*.\end{aligned}$$

Let u be in X with $u \not\equiv 0$. From assumption (J4) and relation (2.10), we obtain that

$$\begin{aligned}\frac{\Phi(u)}{|\Psi(u)|} &= \frac{\int_\Omega \Phi_0(x,\nabla u)\,dx}{\int_\Omega F(x,u)\,dx} \ge \frac{\frac{c_*}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega \frac{|F(x,u)|}{m(x)|u|^{p(x)}} m(x)\,|u|^{p(x)}\,dx}\\ &\ge \frac{\frac{c_*}{p_+}\int_\Omega |\nabla u|^{p(x)}\,dx}{\frac{\mathcal{C}_f}{p_-}\int_\Omega m(x)\,|u|^{p(x)}\,dx} \ge \frac{c_* p_-}{\mathcal{C}_f p_+}\lambda_* \ge \frac{c_*\lambda_* p_-}{(\mathcal{C}_f + |\theta|\,\mathcal{C}_g)p_+} = \ell_*.\end{aligned}$$

Hence we have $\ell^* \ge \ell_*$. Now we claim that there exists a real number r satisfying condition (2.11). For any $u \in \Psi^{-1}((-\infty,0))$, we deduce that

$$\begin{aligned}\chi_1(r) &= \inf_{u\in\Psi^{-1}((-\infty,r))}\frac{\inf_{v\in\Psi^{-1}(r)}\Phi(v)-\Phi(u)}{\Psi(u)-r}\\ &\le \frac{\inf_{v\in\Psi^{-1}(r)}\Phi(v)-\Phi(u)}{\Psi(u)-r} \le \frac{\Phi(u)}{r-\Psi(u)}\end{aligned}$$

for all $r \in (\Psi(u),0)$. This implies that

$$\limsup_{r\to 0-}\chi_1(r) \le -\frac{\Phi(u)}{\Psi(u)}$$

for all $u \in \Psi^{-1}((-\infty,0))$. Hence we have that

$$\limsup_{r\to 0-}\chi_1(r) \le \chi_1(0) = \ell^*.$$

Now we show that there exists a positive real number M_* such that

$$|F(x,s)| \le M_* m(x)\,|s|^{\xi_1(x)} \tag{2.12}$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$. First of all, it follows from (F1) and (F4) that $f(x,0) = 0$ for almost all $x \in \Omega$. In fact, if there exists $A \subset \Omega$, $|A| > 0$ such that $|f(x,0)| > 0$ for all $x \in A$, then $\lim_{s\to 0}\frac{|f(x,s)|}{m(x)|s|^{\xi_1(x)-1}} = \infty$ for all $x \in A$. This contradicts assumption (F4). Thus, we obtain that $\limsup_{s\to 0}\frac{|F(x,s)|}{m(x)|s|^{\xi_1(x)}} < \infty$ uniformly almost all in Ω by the L'Hôpital rule. Let us denote

$$M_3 = \limsup_{s\to 0}\frac{|F(x,s)|}{m(x)\,|s|^{\xi_1(x)}}$$

for almost all $x \in \Omega$. Then there exists $\delta > 0$ such that $|F(x,s)| \le (M_3 + 1)m(x)\,|s|^{\xi_1(x)}$ for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$ with $|s| < \delta$. Next, let s be fixed with $|s| \ge \delta$. It follows from (2.10) that

$$|F(x,s)| \le \frac{\mathcal{C}_f}{p_-}\,|s|^{p(x)-\xi_1(x)}\,m(x)\,|s|^{\xi_1(x)}$$

$$\leq \frac{\mathcal{C}_f(\delta^{p_- - (\xi_1)_+} + \delta^{p_+ - (\xi_1)_-})}{p_-} m(x)\,|s|^{\xi_1(x)}$$

for almost all $x \in \Omega$. Hence the relation (2.12) holds, where

$$M_* = \max\left\{ M_3 + 1, \frac{\mathcal{C}_f(\delta^{p_- - (\xi_1)_+} + \delta^{p_+ - (\xi_1)_-})}{p_-} \right\}.$$

This implies that

$$|\Psi(u)| \leq \int_\Omega M_* m(x)\,|u|^{\xi_1(x)}\,dx \leq \frac{1}{\lambda_* p_+}\|u\|_X^\alpha + 2C_9 M_* \|m\|_{L^\infty(\Omega)}\|u\|_X^\beta$$

for a positive constant C_9 and for all $u \in X$, where α is either p_+ or p_- and β is either $(\xi_1)_+$ or $(\xi_1)_-$. If $r < 0$ and $v \in \Psi^{-1}(r)$, then it follows from assumption (J4) that

$$\begin{aligned} p_+ r &= p_+ \Psi(v) \\ &\geq -\frac{1}{\lambda_*}\|u\|_X^\alpha - 2C_9 M_* p_+ \|m\|_{L^\infty(\Omega)}\|v\|_X^\beta \\ &\geq -\frac{p_+}{c_*\lambda_*}\Phi(v) - 2C_9 M_* p_+ \|m\|_{L^\infty(\Omega)} \left(\frac{p_+}{c_*}\Phi(v)\right)^{\frac{\beta}{\alpha}} \\ &= -\frac{p_+}{c_*\lambda_*}\Phi(v) - 2C_9 M_* \|m\|_{L^\infty(\Omega)} \frac{p_+^{\frac{\beta}{\alpha}+1}}{c_*}\Phi(v)^{\frac{\beta}{\alpha}}. \end{aligned} \tag{2.13}$$

Since $u = 0 \in \Psi^{-1}((r, +\infty))$, we assert

$$\chi_2(r) \geq \frac{1}{|r|} \inf_{v \in \Psi^{-1}(r)} \Phi(v),$$

and hence there exists $u_r \in \Psi^{-1}((r, +\infty))$ such that

$$\Phi(u_r) = \inf_{v \in \Psi^{-1}((r,+\infty))} \Phi(v);$$

see Theorem 6.1.1 in [3]. According to inequality (2.13), we deduce that

$$\begin{aligned} p_+ &\leq \frac{p_+}{c_*\lambda_*}\frac{\Phi(u_r)}{|r|} + \tilde{C}\,|r|^{\frac{\beta}{\alpha}-1}\left(\frac{\Phi(u_r)}{|r|}\right)^{\frac{\beta}{\alpha}} \\ &\leq \frac{p_+}{c_*\lambda_*}\chi_2(r) + \tilde{C}\,|r|^{\frac{\beta}{\alpha}-1}\,\chi_2(r)^{\frac{\beta}{\alpha}}, \end{aligned} \tag{2.14}$$

where a positive constant $\tilde{C}$ is denoted by

$$\tilde{C} = 2C_9 M_* \|m\|_{L^\infty(\Omega)} \frac{p_+^{\frac{\beta}{\alpha}+1}}{c_*}.$$

Then two possibilities are considered; either χ_2 is locally bounded at $0-$ so that inequality (2.14) implies $\liminf_{r\to 0-}\chi_2(r) \geq c_*\lambda_*$ because $\beta > \alpha$ or $\limsup_{r\to 0-}\chi_2(r) = \infty$.

Since the functional $I_{\lambda,\theta} := \Phi(u) + \lambda\,(\Psi(u) + \theta H(u))$ is coercive for all $\lambda, \theta \in \mathbb{R}$, we set $I = \mathbb{R}$. For all integers $n \geq n^* := 1 + 2/[c_*\lambda_* - \ell^*]$, there exists a negative sequence $\{r_n\}$ such that $r_n \to 0$ as $n \to \infty$ with $\chi_1(r_n) < \ell^* + 1/n < c_*\lambda_* - 1/n < \chi_2(r_n)$. In conclusion, since $u \equiv 0$ is a critical point of $I_{\lambda,\theta}$, according to Lemma 2.14, there exits $\tau > 0$ such that problem $(\mathrm{B}_{\lambda,\theta})$ admit at least two distinct weak solutions for each compact interval

$$[a,b] \subset (\ell^*, c_*\lambda_*) = \bigcup_{n=n^*}^{\infty} \left[\ell^* + \frac{1}{n}, c_*\lambda_* - \frac{1}{n}\right] \subset \bigcup_{n=n^*}^{\infty} (\chi_1(r_n),\ \chi_2(r_n))$$

and for every $\lambda \in [a,b]$ and $\theta \in (-\tau,\tau)$. This completes the proof. □

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